1. Weak derivatives

Given a smooth open bounded domain $\Omega \subset \mathbb{R}^n$, we define several sets of functions defined on Ω.

$$
\begin{aligned}
L^p(\Omega) &= \{u : \int_\Omega |u(x)|^p dx < +\infty\},\\
L^p_{\text{loc}}(\Omega) &= \{u : \int_K |u(x)|^p dx < +\infty \text{ for every compact set } K \subset \Omega\},\\
C^\infty(\Omega) &= \{u : u \text{ is smooth in } \Omega\},\\
C^\infty_c(\Omega) &= \{u \in C^\infty(\Omega) : u \text{ is compactly supported in } \Omega\},
\end{aligned}
$$

namely $u \in C^\infty_c(\Omega)$ implies that there exists a compact set $K_u \in \Omega$ such that $u = 0$ on $\Omega \backslash K_u$. (Note that since Ω is open, $K_u \cap \partial\Omega = \emptyset$.)

Suppose that $u, v \in L^1_{\text{loc}}(\Omega)$ satisfy

$$
\int_\Omega u \nabla_k \varphi dx = -\int_\Omega v \varphi dx, \tag{1}
$$

for all $\varphi \in C^\infty_c(\Omega)$. Then, we say that v is a weak derivative of u, and we denote by $v = \nabla_k u$.

We define

$$
\begin{aligned}
W^{1,p}(\Omega) &= \{u \in L^p(\Omega) : u \text{ has a weak derivative } \nabla_k u \in L^p(\Omega) \text{ for each } k = 1, \cdots, n\},\\
W^{1,p}_0(\Omega) &= \{u \in W^{1,p}(\Omega) : \text{there exists a sequence } \ u_m \in C^\infty_c(\Omega) \text{ such that } \lim_{m\to\infty} \|u - u_m\|_{W^{1,p}} = 0.\},
\end{aligned}
$$

where

$$
\|v\|_{W^{1,p}} = \int_\Omega |v|^p + \sum_{k=1}^n |\nabla_k v|^p dx.
$$

In particular, we define

$$
H^1(\Omega) = W^{1,2}(\Omega), \qquad H^1_0(\Omega) = W^{1,2}_0(\Omega), \qquad \|\cdot\|_{W^{1,2}} = \|\cdot\|_{H^1}.
$$

Moreover, we define inner product

$$
\langle u, v\rangle_{L^2} = \int_\Omega uv dx, \qquad \langle u, v\rangle_{H^1} = \int_\Omega uv + \nabla u \cdot \nabla v dx,
$$

where $\nabla u \cdot \nabla v = \sum_{k=1}^n \nabla_k u \nabla_k v$.

2. Functionals

We call a mapping from a space X (of functions) into $\mathbb{R}$ as a functional. For example, the delta function is a functional. To be specific, for the space of function $X = \{f : \mathbb{R} \to \mathbb{R}\}$ the delta function δ maps X into $\mathbb{R}$ by $\delta_0(f) = \int_{\mathbb{R}} \delta(0) f(x) dx = f(0)$.

We denote by $H^{-1}(\Omega)$ the set of continuous linear functionals defined on $H_0^1(\Omega)$.

Example. Given $u \in H_0^1(\Omega)$, we can define a functional $F_u \in H^{-1}$ by

$$F_u(v) = \langle u, v \rangle_{H^1}.$$

3. Weak solutions

Given a function $f \in L^2(\Omega)$, we say that $u \in H_0^1(\Omega)$ is a weak solution to the Dirichlet problem

$$\begin{aligned} \Delta u = f \qquad & \text{in} \quad \Omega, \\ u = 0, \qquad & \text{on} \quad \partial\Omega, \end{aligned} \tag{2}$$

if the following holds for all $v \in H_0^1(\Omega)$

$$\int_\Omega \nabla u \cdot \nabla v + f v = 0.$$

4. Interior regularity of classical solutions

We introduce the Poincaré inequality.

Theorem 1 (Poincaré)**.** *There exists some constant C depending on the bounded Ω such that*

$$\|u\|_{L^2} \leqslant C \|\nabla u\|_{L^2}, \tag{3}$$

holds for all $u \in H_0^1(\Omega)$.

Also, we can use the Sobolev inequality.

Theorem 2. *There exists some constant C depending on p, q, n and the bounded Ω such that if $p < n$*

$$\|u\|_{L^q} \leqslant C\|\nabla u\|_{L^p}, \tag{4}$$

holds for all $u \in W_0^{1,p}(\Omega)$ and $q \leqslant \frac{np}{n-p}$.

If $p > n$, then

$$\sup_\Omega |u| \leqslant C\|\nabla u\|_{L^p}, \tag{5}$$

holds for all $u \in W_0^{1,p}(\Omega)$

Theorem 3. *Suppose that $u \in C^\infty(\Omega)$ solves (2) for $f \in C^\infty(\Omega)$. Then, for each compact set $K \subset \Omega$, there exists some constant C depending on K, Ω, f such that*

$$\|u\|_{H^1(K)} \leqslant C\|f\|_{L^2(\Omega)}. \tag{6}$$

Proof. We begin by choosing a cut-off function $\eta \in C_c^\infty(\Omega)$ such that $\eta = 1$ on K.

$$\int_\Omega \eta^2|\nabla u|^2 dx = -\int_\Omega \eta^2 u\Delta u + 2\eta u\nabla u\nabla\eta dx \leqslant \int_\Omega -\eta^2 uf + \tfrac{1}{2}\eta^2|\nabla u|^2 + 2u^2|\nabla\eta|dx.$$

namely,

$$\frac{1}{2}\int_K |\nabla u|^2 dx \leqslant \frac{1}{2}\int_\Omega \eta^2|\nabla u|^2 dx \leqslant \int_\Omega -\eta^2 uf + 2u^2|\nabla\eta|^2 dx \leqslant C\int_\Omega u^2 + f^2 dx.$$

On the other hand, the Poincaré inequality yields

$$\int_\Omega u^2 dx \leqslant C\int_\Omega |\nabla u|^2 dx \leqslant -C\int_\Omega fudx \leqslant C\int_\Omega \epsilon u^2 + \tfrac{1}{2\epsilon}f^2 dx.$$

Choosing ϵ small enough, for some large C we have

$$\frac{1}{2}\int_\Omega u^2 dx \leqslant C\int_\Omega f^2 dx.$$

Therefore,

$$\frac{1}{2}\int_K |\nabla u|^2 dx + \frac{1}{2}\int_\Omega u^2 dx \leqslant C\int_\Omega u^2 + f^2 dx \leqslant C\int_\Omega f^2 dx. \tag{7}$$

□